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Rapid Calculation for Convective Heat Transfer in Laminar Boundary Layer by Gaussian Quadratures

D. R. JENG AND R. S. REDDY GORLA

Department of Mechanical Engineering, The University of Toledo, Toledo, Ohio (U.S.A.)

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SUMMARY

The integral of the form $\int_0^\infty \exp\left[-Pr\int_0^\eta f(\zeta)d\zeta\right]d\eta$, which arises in the convective heat transfer with constant wall temperature, is integrated by using Gauss–Laguerre and Gauss–Legendre Quadrature formulae. It is shown that the Nusselt number can be expressed explicitly in terms of the Prandtl number and the method proposed in this paper is valid for wide range of Prandtl numbers. Examples are given for the cases of flow over a semi-infinite plate and two-dimensional and axisymmetrical stagnations. The results are compared with the exact solutions for Prandtl numbers ranging from 0.006 to 100 (flat plate) and 0.01 to 50 (two-dimensional and axisymmetrical stagnation flows).

1. Introduction

The theory of boundary layers and convective heat transfer in boundary layers has grown considerably since Prandtl introduced the concept of boundary layers in 1904. However, there was not a general method of solution for the velocity boundary-layer, and hence for the thermal boundary-layer equations, because the velocity field is a base for obtaining temperature field. There are several exact and approximate methods known in the literature [1, 2] for the calculation of both the wall friction and the heat transfer.

The exact solution method for the convective heat transfer problems is usually to find a coordinate transformation to transform momentum and energy equations to ordinary or partial differential equations and then simultaneously solve by numerical procedure. The presently existing numerical procedure is laborious and it can only be used for a given Prandtl number; it results in empirical laws which have no physical foundation. In order to avoid the lengthy exact calculations, several approximate methods have been developed. One of the best known approximate methods is that of Karman–Pohlhausen. Generally, this method is still troublesome and its accuracy is disappointing especially when applied to the heat transfer problems. Merk [3] and Meksyn [4] developed an asymptotic method which is capable of expressing the Nusselt number explicitly in terms of a negative power of Prandtl number in series form. For small value of Prandtl number (Pr), the series may become divergent and Euler's procedure for the evaluation of the series may be used, however, the result is not satisfactory when $Pr < 0.1$.

In this paper, a rapid method of calculating the convective heat transfer by means of Gaussian quadrature formulae is proposed. In what follows, we limit the consideration to the case of a steady, laminar incompressible flow with constant properties and negligible dissipation. Furthermore, the free stream and wall temperatures are assumed to be constant. Under such conditions, in many cases of convective heat transfer, the problem is reduced to the evaluation of an integral

$$\int_0^\infty \exp\left[-Pr\int_0^\eta f(\zeta)d\zeta\right]d\eta\,, \tag{1}$$

where ζ is a dummy variable and $f(\eta)$ is a solution of the transformed momentum equation concerned. Let's consider a specific problem, for example, the heat transfer from a two-dimensional or axisymmetrical body immersed in a laminar, incompressible stream. For these cases we may introduce the coordinates (x, y), x being the distance from the forward stagnation point measured along the circumference of the two-dimensional profile or median line of the axisymmetrical body, and y being the normal distance from the wall of the body. The velocity components in x- and y-directions are denoted by u and v. For axisymmetrical bodies, we also introduce r, the distance from a surface element of the body to the axis of symmetry.

If one introduces:

the coordinate transformation:

$$\left.\begin{array}{l} x \\ \\ y \end{array}\right\} \rightarrow \left\{\begin{array}{l} \xi = \displaystyle\int_0^x \frac{U_e(x)}{U_\infty}\left(\frac{r}{L}\right)^{2i} dx \\ \\ \eta = \displaystyle\frac{1}{(2\nu U_\infty \xi)^{\frac{1}{2}}} U_e\left(\frac{r}{L}\right)^i y\,, \end{array}\right. \tag{2}$$

the stream function:

$$\psi = (2\xi U_\infty \nu)^{\frac{1}{2}} f(\xi, \eta) \tag{3a}$$

such that

$$u = \left(\frac{L}{r}\right)^i \frac{\partial\psi}{\partial y}\,, \quad v = -\left(\frac{L}{r}\right)^i \frac{\partial\psi}{\partial x}\,, \tag{3b}$$

the dimensionless temperature:

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{4}$$

with $i = 0, 1$, respectively, for two-dimensional and axisymmetrical flows, L the characteristic length, U_e the velocity at the edge of the boundary layer, U_∞ free stream velocity, ν the kinematic viscosity, T_∞ and T_w respectively, the free stream and wall temperatures into the momentum and energy equations and further apply the Meksyn's wedge approximation (zeroth order solution for the transformed momentum and energy equations) originally proposed by Merk [4], the local Nusselt number may be written by

$$Nu_\xi = \frac{q_w \xi}{k(T_w - T_\infty)} = \frac{U_e}{U_\infty}\frac{1}{\sqrt{2}}\left[-\theta_0'(o)\right] \cdot Re_\xi^{\frac{1}{2}} \left(\frac{r}{L}\right)^i \tag{5}$$

where Reynolds number $Re_\xi = U_\infty \xi/\nu$.

The quantity, $-\theta'(o)$, is the negative dimensionless temperature gradient at the wall and is defined by

$$-\theta'(o) = \left[\int_0^\infty \exp\left(-Pr\int_0^\eta f(\zeta)d\zeta\right)d\eta\right]^{-1} \tag{6}$$

in which f satisfies

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{7}$$

with boundary conditions

$$f = f' = 0 \quad \eta = 0 \text{ and } f' = 1 \qquad \eta \to \infty\,. \tag{8}$$

In the above, the primes designate derivatives with respect to η while f is a function of η only and β is the "wedge variable" defined by

$$\beta = \frac{2\xi}{U_e}\frac{dU_e}{d\xi}\,. \tag{9}$$

The derivation of (5), (6) and (7) may be found in refs. [1, 3, 4] and will not be repeated here. For a flow past a wedge, the velocity at the edge of the boundary layer is given by $U_e(x) = Cx^m$, where C and m are constants. Under this condition, β becomes a constant and (6) and (7) yield exact results. Equation (7) has been solved by Howarth [5] for $\beta=0$, Hiemenz [6] for $\beta=1$, Frössling [7] for $\beta=\frac{1}{2}$ and by Hartree [8] for β ranging from -0.198 to 2.4. Their results are summarized in refs. [1, 2]. The value of $f(\eta)$ together with f' and f'' * is given by a table of numerical values which f, f' and f'' assume for a discrete set of their arguments. Equation (6) has been calculated numerically by Eckert [9], Pohlhausen [10] for some specified value of β and Pr. Asymptotic expansion of (6) was derived by Merk [3] using the method of steepest descent. In this paper, an entirely new approximate solution method is developed for the numerical integration of (6) by means of Gaussian quadrature formulae. The method is simple and the numerical calculation is rapid and yet the method will provide the highest accuracy possible.

2. Analysis

Gaussian Quadrature

In the convective heat transfer with constant wall temperature, we are primarily interested in evaluating the surface heat flux. Thus, the quantities desired are of equation (6) and the question as to the best quadrature formula for the numerical evaluation of the integral of the form $\int_0^\infty \Phi(\eta)d\eta$ is therefore of considerable practical interest. In this section, we like to discuss how we can develop the quadrature for such a purpose.

Let's consider quite generally the integral

$$\int_a^b W(\eta)\Phi(\eta)d\eta \tag{10}$$

between two assigned limits a and b of a continuous function $\Phi(\eta)$ weighted by some known function $W(\eta)$. Suppose that $\Phi(\eta)$ stands for a more general type of a function whose primitive function may not be easily ascertained by the method of formal analysis and under such circumstances analytical integration of (10) proves of little hope; moreover, in the convective heat transfer, $\Phi(\eta)$ is not known in its analytical form but merely specified by a table of numerical values at a discrete set of its arguments of η's, and the indefinite integral methods become altogether inapplicable. However, the integration of (10) may be approximately achieved by numerical quadrature formulae.

According to the Weierstrass' theorem on polynomial approximation, any arbitrary continuous function defined in a finite interval $a \leqq \eta \leqq b$ can be approximated over the whole interval (a, b) to any accuracy by a polynomial of sufficiently high degree [11]. Let's now interpolate the function $\Phi(\eta)$ which appears in the integral (10) by a polynomial of $(2n-1)$ degree and the remainder term, if $\Phi(\eta)$ is of a degree higher than $(2n-1)$. Using Hermite's interpolation formula for polynomial, we write

$$\Phi(\eta) = \sum_{j=1}^{n} h_j(\eta)\Phi(\eta_j) + \sum_{j=1}^{n} g_j(\eta)\Phi^{(1)}(\eta_j) + \frac{\Phi^{(2n)}(\lambda)}{(2n)!}[P_n(\eta)]^2 \tag{11a**}$$

where

$$h_j(\eta) = \left[1-(\eta-\eta_j)\frac{P_n^{(2)}(\eta_j)}{P_n^{(1)}(\eta_j)}\right][l_j(\eta)]^2 \tag{11b}$$

$$g_j(\eta) = (\eta-\eta_j)[l_j(\eta)]^2 \tag{11c}$$

$$l_j(\eta) = \frac{P_n(\eta)}{(\eta-\eta_j)P_n^{(1)}(\eta_j)} \tag{11d}$$

$P_n(\eta)$ is a polynomial of degree n abbreviated by

* Only for $\beta=0$, $\beta=\frac{1}{2}$ and $\beta=1$. For other values of β, Hartree gave only the value $f'(\eta)$ and $f''(0)$. However, $f(\eta)$ and $f''(\eta)$ may be calculated by Taylor's series.

** The superscript in the parenthesis denotes order of the derivative.

$$P_n(\eta) = \prod_{j=1}^{n} (\eta - \eta_j) \tag{11e}$$

and

$$P_n^{(1)}(\eta_j) = \left(\frac{dP_n}{d\eta}\right)_{\eta=\eta_j} = \prod_{i \neq j} (\eta_j - \eta_i) . \tag{11f}$$

In the above, the η_j's denote n discrete abscissae at which the polynomial will take the same value of $\Phi(\eta)$ and $\Phi^{(1)}(\eta)$ as Φ does.

When $\Phi(\eta)$ is a polynomial of degree $(2n-1)$ or less, $\Phi^{(2n)}(\eta)=0$ and the remainder term on the right-hand side of equation (11a) vanishes. The value λ appearing in the remainder term denotes some value of the variable within the interval $a<\lambda<b$ and will, however, vary with η. Actually, λ is merely defined in such a way that to every value of an interpolated η, there corresponds at least one value of λ such that $a<\eta<b$, which renders equation (11a) exact. It should be kept in mind that we are, of course, unable to determine the actual values of $\Phi^{(2n)}(\lambda)$ since the function $\Phi(\eta)$ is unknown.

To evaluate the numerical integral of (10), we substitute (11) into (10) and obtain

$$\int_a^b W(\eta)\Phi(\eta)d\eta = \sum_{j=1}^{n} H_j\Phi(\eta_j) + \sum_{j=1}^{n} G_j\Phi^{(1)}(\eta_j) + \int_a^b W\frac{\Phi^{(2n)}(\lambda)}{(2n)!}[P_n(\eta)]^2 d\eta \tag{12a}$$

where

$$H_j = \int_a^b Wh_j(\eta)d\eta = \int_a^b Wl_j^2(\eta)d\eta - \frac{P_n^{(2)}(\eta_j)}{P_n^{(1)}(\eta_j)}\int_a^b Wl_j^2(\eta)(\eta-\eta_j)d\eta \tag{12b}$$

$$G_j = \int_a^b Wg_j(\eta)d\eta = \int_a^b Wl_j^2(\eta)(\eta-\eta_j)d\eta . \tag{12c}$$

Thus, (10) may be approximately integrated by neglecting the last term (error term) in (12a). If $\Phi(\eta)$ is of degree not in excess of $(2n-1)$, (12a) is exact without the error term. There are n abscissae, namely η_j $(j=1, 2, \ldots, n)$, remaining to be determined. In the method of Newton–Cotes, all the abscissae η_j are equally spaced within the interval of integration, but Gauss showed that it would be advantageous to choose η_j differently. He has shown that if η_j's are chosen as roots of such an orthogonal polynomial $P_n(\eta)$ satisfying the following orthogonality conditions [11],

$$\int_a^b P_n(\eta)W(\eta)\eta^l d\eta = 0 \qquad (l=0, 1, \ldots, n-1) \tag{13}$$

then $G_j=0$ and H_j reduces to

$$H_j = \int_a^b Wl_j^2(\eta)d\eta = \int_a^b Wl_j(\eta)d\eta . \tag{14}$$

Another advantage of determining $P_n(\eta)$ by the orthogonality condition (13) and defining η_j's as the zeros of $P_n(\eta)$ is minimization of error term of the corresponding quadrature formula. It is thus clear that Gaussian determination of $P_n(\eta)$ and η_j's indeed plays a double roll, namely, it eliminates one half of the $2n$ terms of the right-hand side of (12a), thus enabling us to express the integral of a polynomial $\Phi(\eta)$ of degree $(2n-1)$ exactly as a weighted mean of its n particular values $\Phi(\eta_j)$, and at the same time assures us that error committed by neglecting the error term will be a minimum. By neglecting the error term, an approximate evaluation of the integral is then obtained by the formula

$$\int_a^b W(\eta)\Phi(\eta) = \sum_{j=1}^{n} H_j\Phi(\eta_j) . \tag{15}$$

In the previous analysis, so far we have assumed both limits of integration (a, b) to be finite numbers, however, by appropriate selection of the weighted function $W(\eta)$, the theory and equation will continue to apply for infinite limits.

Solution for a Flat Plate

This is a particular case of problems with $\beta=0$ in (7). It can, however, be treated simpler as a separate problem; and in view of its importance, it will be considered in detail. In this case, we have $i=0$, $U_e=U_\infty$, $\xi=x$. We further modify the coordinate η and stream function ψ by a factor of $2^{\frac{1}{2}}$ so that the numerical table prepared by Howarth [2, 5] is at our disposal for heat transfer calculation.

We let

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \tag{16}$$

and

$$\psi = (U_\infty x\nu)^{\frac{1}{2}} f(x, \eta)\,. \tag{17}$$

Hence, (6) becomes

$$-\theta'(o) = \left[\int_0^\infty \exp\left(-\frac{Pr}{2}\int_0^\eta f\, d\zeta\right) d\eta\right]^{-1} \tag{18}$$

in which f satisfies

$$f''' + \tfrac{1}{2} f f'' = 0 \tag{19}$$

with boundary conditions

$$f = f' = 0 \text{ at } \eta = 0, \quad f' = 1 \text{ as } \eta \to \infty\,.$$

If one denotes the Reynolds number Re_x by $U_\infty x/\nu$ and Nusselt number Nu by $q_w x/\{k(T_w - T_\infty)\}$, then

$$Nu\, Re_x^{-\frac{1}{2}} = -\theta'(o)\,. \tag{20}$$

Averaging the Nusselt number over the length L of the plate, we get

$$\overline{Nu}\, Re_L^{-\frac{1}{2}} = -2\theta'(o) \tag{21}$$

where $Re_L = U_\infty L/\nu$. Equation (18) was originally integrated numerically by Pohlhausen [10]. In this section, we like to demonstrate how well and easily the application of (15) to evaluate (18) can be practically done. For that purpose, we write

$$\int_0^\infty \exp\left(-\frac{Pr}{2}\int_0^\eta f\, d\zeta\right) d\eta = \int_0^\infty \exp(-\eta)\exp\left[\eta + Pr \ln\frac{f''(\eta)}{f''(o)}\right] d\eta \tag{22}$$

Since

$$\int_0^\eta f\, d\zeta = -2\ln\frac{f''(\eta)}{f''(o)}\,.$$

To evaluate (22) by using (15), we let

$$\int_0^\infty e^{-\eta}\Phi(\eta)\, d\eta = \sum_{j=1}^{n} H_j \Phi(\eta_j) \tag{23}$$

where

$$W(\eta) = e^{-\eta} \tag{24}$$

and

$$\Phi(\eta_j) = \exp\left[\eta_j + Pr \ln\frac{f''(\eta_j)}{f''(o)}\right] \tag{25}$$

The orthogonal polynomial satisfying (13) with $a=0$, $b=\infty$ and $W(\eta)=e^{-\eta}$ is a Laguerre polynomial defined in ref. [11] and can be written as:

$$P_n(\eta) = \frac{e^\eta}{n!}\frac{d^n}{d\eta^n}(e^{-\eta}\eta^n) \tag{26}$$

and hence, H_j becomes

$$H_j = \frac{1}{\eta_j}[P_n^{(1)}(\eta_j)]^{-2}. \tag{27}$$

It follows that (18) becomes

$$-\theta'(o) = \frac{[f''(o)]^{Pr}}{\sum_{j=1}^{n} H_j e^{\eta_j}[f''(\eta_j)]^{Pr}} \tag{28}$$

in which the η_j's ($j=1, 2, \ldots, n$) are the roots of Laguerre polynomials. The values of η_j together with the corresponding H_j are calculated by Salzer and Zucker [12]*. The results of $f''(\eta)$ calculated by Howarth are tabulated for $\eta = 0\ (0.2)\ 8.8$ in ref. [2] and $f''(\eta_j)$ may not be known to us directly from the table. However, its value may be obtained by means of Taylor's extrapolation formula

$$f''(\eta_j) = \sum_{n=2}^{\infty} \frac{f^{(n)}(\eta_0)}{(n-2)!}(\eta_j - \eta_0)^{n-2} \tag{29}$$

with

$$f^{(3)} = -\tfrac{1}{2}ff^{(2)}, \quad f^{(4)} = -\tfrac{1}{2}[ff^{(3)} + f^{(1)}f^{(2)}]$$

$$f^{(5)} = -\tfrac{1}{2}[ff^{(4)} + 2f^{(1)}f^{(3)} + (f^{(2)^2})]$$

$$f^{(6)} = -\tfrac{1}{2}[ff^{(5)} + 3f^{(1)}f^{(4)} + 4f^{(2)}f^{(3)}] \quad \text{etc.},$$

where η_0 is the closest possible abscissa to η_j given in the table ($\eta_0 < \eta_j$). For large value of η_j (say $\eta_j > 7$), the asymptotic equation

$$f''(\eta_j) = \gamma \exp[-\tfrac{1}{4}(\eta_j - \beta)^2] \tag{30}$$

with $\gamma = 0.231$ and $\beta = 1.73$ may be used for advantage to evaluate $f''(\eta_j)$. Values of H_j and $f''(\eta_j)$ for the corresponding η_j are listed in Table 1 for the six values of n shown.

$-\theta'(o)$ calculated by means of (28) using $n = 1, 2, 3, 4, 5$ and 15 are listed in Table 2 for Prandtl numbers ranging from 0.006 to 100. The results may be compared with the exact values available in the literature.

It can be seen how well the lower orders of approximation represent the exact solution. For $0.1 \leqq Pr \leqq 10$, four terms ($n=4$) in (28) are sufficient if the desired accuracy does not exceed 2%. Using 5 terms, the error is less than 0.7% for $0.1 \leqq Pr \leqq 1.1$. Equation (28) with 15 terms leads to surprisingly good results even to the fourth decimal place for $0.6 \leqq Pr \leqq 1.1$, and for $0.006 \leqq Pr \leqq 0.5$ the error is less than 0.3% while for $1.1 < Pr \leqq 15$ the error is about 0.4%. The deviation becomes slightly large as Pr increases. The reason for this is that when Pr becomes large, thickness of the thermal boundary layer is thin and is only a small fraction of the momentum boundary layer thickness. Thus, the main contribution to the integral of (22) comes from the line integral along the real axis η at the vicinity of the wall ($\eta = 0$). This phenomenon may also be clearly seen from (28). Although we have employed 15 terms in the calculation, only the first few terms containing small η_j contribute to the summation when Prandtl number becomes large. Thus, for large Prandtl numbers, a higher order approximation is required if a more accurate result is desired.

* Their data are also reproduced in ref. [11].

TABLE 1

*Value of η_j, H_j and $f''(\eta_j)$ in the equation (28) and $F(\eta_j)$ in the equation (31) for the calculation of Nusselt number**

	η_j	H_j	$f''(\eta_j)$	$F(\eta_j)$	
				$\beta=1$ (2 Dim. stagnation)	$\beta=\frac{1}{2}$ (Axi-symmetrical)
$n=1$	1	1	0.32301		
$n=2$	0.58579	0.85355	0.33022		
	3.41421	0.14645	0.11642		
$n=3$	0.41577	0.71109	0.33140		
	2.29428	0.27852	0.23900		
	6.28995	0.(1)10389	0.(2)12765		
$n=4$	0.32255	0.60315	0.33176		
	1.74576	0.35742	0.28691		
	4.53662	0.(1)38888	0.(1)32276		
	9.39507	0.(3)53929	Eq. (30)		
$n=5$	0.26356	0.52176	0.33190		
	1.41340	0.39867	0.30719		
	3.59643	0.(1)75942	0.(1)98427		
	7.08581	0.(2)36118	↑ Eq. (30)		
	12.64080	0.(4)23370	↓		
$n=15$	0.(1)93308	0.21823	0.33205	0.(3)16372	0.(3)12404
	0.49269	0.34221	0.33096	0.(1)22141	0.(1)17273
	1.21560	0.26303	0.31600	0.28311	0.23259
	2.26995	0.12643	0.24145	1.46368	1.27702
	3.66762	0.(1)40207	0.(1)91798	4.70899	4.30642
	5.42534	0.(2)85637	0.(2)75676	11.56649	10.88359
	7.56592	0.(2)12124	↑	24.09730	23.06575
	10.12023	0.(3)11167			45.04032	43.59847
	13.13029	0.(5)64599			78.08613	76.16927
	16.65441	0.(6)22263			128.28209	125.81624
	20.77648	0.(8)42274	Eq. (30)	202.75360	199.64589
	25.62389	0.(10)39219			312.09061	308.20675
	31.40752	0.(12)14565			473.26244	468.47721
	38.53068	0.(15)14830			717.74434	711.83672
	48.02609	0.(19)16006	↓	1122.53610	1115.14200

* (a) η_j and H_j were originally calculated to twelve significant decimal figures. We have truncated them to five.
(b) The number in the parentheses stands for the number of zeros between the decimal point and the first significant figure.
(c) The following asymptotic equations for $f(\zeta)$ were used in the calculation of $F(\eta_j)$ in (37) for large ζ:

$\beta=\frac{1}{2}$ $f(\zeta)=\zeta+0.393\exp(-\frac{1}{2}\zeta^2)(\zeta^{-3}-6\zeta^{-5}+45\zeta^{-7}+\ldots)$ $\quad \zeta>4.4$

$\beta=1$ $f(\zeta)=\zeta+0.645\exp(-\frac{1}{2}\zeta^2)(\frac{1}{2}\zeta^{-2}-6\zeta^{-6}+\ldots)$ $\quad \zeta>4.4$

Solution for Arbitrary Values of β

In this section, we shall demonstrate that by the proposed method, one can also calculate the Nusselt number for $\beta\neq 0$, i.e. for boundary layers with longitudinal pressure gradient. To this end, applying Gauss–Laguerre quadrature, we write (6) as

$$-\theta'(o)=\left\{\sum_{j=1}^{n} H_j \exp[\eta_j - Pr\, F(\eta_j)]\right\}^{-1} \tag{31}$$

where

TABLE 2

Comparison of results for $-\theta'(o)$ for flat plate

Pr \ n	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=15$	Exact*
0.006	0.3679	0.1679	0.1081	0.08068	0.06971	0.04084	0.04073
0.01	0.3680	0.1684	0.1095	0.08325	0.07341	0.05161	0.05159
0.1	0.3689	0.1805	0.1437	0.1395	0.1410	0.1393	0.1381
0.5	0.3730	0.2401	0.2656	0.2583	0.2589	0.2593	0.2591
0.6	0.3740	0.2563	0.2824	0.2773	0.2759	0.2769	0.2769
0.7	0.3751	0.2729	0.2955	0.2948	0.2911	0.2927	0.2927
0.8	0.3761	0.2898	0.3064	0.3109	0.3053	0.3069	0.3069
0.9	0.3771	0.3069	0.3157	0.3254	0.3186	0.3200	0.3200
1.0	0.3782	0.3241	0.3242	0.3385	0.3314	0.3320	0.3320
1.1	0.3792	0.3414	0.3321	0.3502	0.3435	0.3434	0.3434
7.0	0.4464	0.6768	0.7467	0.6395	0.6144	0.6471	0.6459
10	0.4850	0.6895	0.8624	0.7691	0.7001	0.7289	0.7282
15	0.5568	0.7091	0.9382	0.9522	0.8445	0.8305	0.8341
50	1.4646	0.8623	1.0250	1.2542	1.4359	1.2691	1.2472
100	5.8312	1.1401	1.1322	1.3144	1.5436	1.5456	1.5718

* The exact numerical results are obtained from the following references for the Prandtl numbers indicated.

Pr from 0.6 to 15, Originally calculated by E. Pohlhausen [10] to three figures and recalculated by Merk [3] to four figures.
$Pr=0.1$ and 0.5, from ref. [3].
$Pr=0.006$ and 0.01, from ref. [13].
$Pr=50$ and 100, Calculated by us using the asymptotic solution given in ref. [3].

$$F(\eta_j) = \int_0^{\eta_j} f(\zeta)d\zeta \,. \tag{32}$$

Equation (32) can not be integrated in its analytical form when $\beta \neq 0$. Here, we propose to evaluate (32) by using Gauss–Legendre quadrature. Let's introduce a transformation:

$$\chi = \frac{2\zeta - \eta_j}{\eta_j} \tag{33}$$

Therefore,

$$\int_0^{\eta_j} f(\zeta)d\zeta = \frac{\eta_j}{2}\int_{-1}^{1} \bar{f}(\chi)d\chi \,. \tag{34}$$

Comparing the integral

$$\int_{-1}^{1} \bar{f}(\chi)d\chi$$

with (10), one finds $a=-1$, $b=1$ and $W(\chi)=1$ and the orthogonal polynomial satisfying (13) under these conditions is the Legendre polynomial of degree m, (to avoid the confusion in the following analysis, we replace n in (13) by m) defined by

$$P_m(\chi) = \frac{1}{2^m m!}\frac{d^m}{d\chi^m}(\chi^2-1)^m \tag{35}$$

and the corresponding weight coefficient H_i reduces to the form

$$H_i = \frac{2}{(1-\chi_i^2)[P_m^{(1)}(\chi_i)]^2} \tag{36}$$

in which χ_i are roots of the Legendre polynomials $P_m(\chi)$. The χ_i and the corresponding values of H_i were calculated by Lowan, Davids and Levenson [14] for $m=2(1)16$ to fifteen decimal places. (Their results will also be found in ref. [11]). For convenience, we reproduce their value

TABLE 3

Roots and weight coefficients of the Gaussian–Legendre quadrature formula to be used in (37)

$m=16$

$\pm\chi_i$	H_i
0.98940	0.027152
0.94458	0.062254
0.86563	0.095159
0.75540	0.12463
0.61788	0.14960
0.45802	0.16916
0.28160	0.18260
0.095013	0.18945

for $m=16$ in Table 3 to five significant decimal figures, since these data will be used in our numerical calculation later. Equation (32) can be re-written as

$$F(\eta_j) = \frac{\eta_j}{2}\sum_{i=1}^{m} H_i\bar{f}(\chi_i) = \frac{\eta_j}{2}\sum_{i=1}^{m} H_i f(\zeta_i) \tag{37}$$

with

$$\zeta_i = \frac{\eta_j(\chi_i+1)}{2}.$$

The value of $f(\zeta_i)$ may not be known to us directly from the table since f is usually tabulated for a discrete set of its abscissa. However, $f(\zeta_i)$ may be calculated by means of Taylor's extrapolation formula

$$f(\zeta_i) = \sum_{n=0}^{\infty}\frac{f^{(n)}(\zeta_0)}{n!}(\zeta_i-\zeta_0)^n \tag{38}$$

with

$$f^{(3)}(\zeta_0) = -f^{(0)}f^{(2)}-\beta[1-(f^{(1)})^2]$$

$$f^{(4)}(\zeta_0) = -f^{(0)}f^{(3)}-f^{(1)}f^{(2)}+2\beta f^{(1)}f^{(2)}$$

$$f^{(5)}(\zeta_0) = -f^{(0)}f^{(4)}+2(\beta-1)f^{(1)}f^{(3)}+(2\beta-1)(f^{(2)})^2$$

$$f^{(6)}(\zeta_0) = -f^{(0)}f^{(5)}+(2\beta-3)f^{(1)}f^{(4)}+2(3\beta-2)f^{(2)}f^{(3)}, \text{ etc.}$$

where ζ_0 is the closest possible abscissa to be given in the table ($\zeta_0<\zeta_i$). After substituting the known values of $f(\zeta_0)$, $f^{(1)}(\zeta_0)$ and $f^{(2)}(\zeta_0)$ into (38), the values of $f^{(n)}(\zeta_0)$ and $f(\zeta_i)$ may readily be obtained. When any ζ_i becomes large, the asymptotic equation for $f(\zeta)$ may be used. If η_j is small, the following alternative formula may also be used for evaluating $F(\eta_j)$,

$$F(\eta_j) = \eta_j^3\sum_{n=0}^{\infty}\frac{f^{(n+2)}(o)}{(n+3)!}\eta_j^n \tag{39}$$

with

$$f^{(2)}(o) = c \quad f^{(3)}(o) = -\beta, \quad f^{(4)}(o) = 0$$
$$f^{(5)}(o) = c^2(2\beta - 1), \quad f^{(6)}(o) = 2c\beta(2 - 3\beta), \text{ etc.}$$

Numerical values of $F(\eta_j)$ using (37) with $m = 16$ were calculated by us and listed in Table 1 for $\beta = 1$ (two-dimensional front stagnation point) and $\beta = \frac{1}{2}$ (axisymmetrical front stagnation point) along with the corresponding η_j for $n = 15$. With $F(\eta_j)$ known, $-\theta'(o)$ can be determined from (31) for any desired value of Prandtl number. The results are shown in Table 4. For the

TABLE 4

Comparison of results for $-\theta'(o)$ for $\beta = \frac{1}{2}$ and 1

Pr		0.01	0.1	0.7	1	10	50
$\beta = 1$	Exact	0.07598	0.2195	0.4959	0.5704	1.3389	2.3533
	from (31) with $n = 15$	0.07598	0.2195	0.4959	0.5704	1.3507 (+.9%)	2.3654 (+.6%)
$\beta = \frac{1}{2}$	Exact	0.07512	0.2132	0.4705	0.5389	1.2389	2.1607
	from (31) with $n = 15$	0.07512	0.2132	0.4705	0.5389	1.2544 (+1.2%)	2.1086 (−2.4%)

purpose of comparison, included are the exact numerical solutions computed by us using electronic computer. It appears that our equation (31) with $n = 15$ leads to satisfactory agreement with the exact value even to the fourth decimal place for Prandtl numbers ranging from 0.01 to 1. The accuracy seems decreasing as Pr increases. The reason for this phenomena is explained in the previous section, and if a more accurate result is desired, the higher order terms in (31) are required.

Concluding Remarks

As has been demonstrated in the text, by applying the Gaussian quadratures, Nusselt number may be expressed explicitly in terms of the Prandtl number. A great advantage of the present approximation method is that no physical hypothesis is needed; the simplifications introduced are of mathematical nature. The calculations may be refined in a straightforward way by considering more terms in the quadrature formula to any desired degree of accuracy. It may be expected that successive refinements converge towards the exact solution. However, for large Prandtl number (say $Pr > 10$), its convergence is rather slow. The lower order approximation ($n = 4$ or 5) will provide a rapid method of obtaining the approximate solution for engineering heat transfer problems.

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